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A general approach to potentials with two known levels

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Abstract

We present the general form of potentials with two given energy levels E_1 and E_2 and find the corresponding wavefunctions. These entities are expressed in terms of one function $\xi(x)$ and one parameter $\Delta E = E_2 - E_1$. We show how the quantum numbers of both levels depend on the properties of the function $\xi(x)$. Our approach does not require one to resort to the technique of supersymmetric quantum mechanics but generates the expression for the superpotential automatically.

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1. Introduction

Potentials whose spectrum can be found exactly are very rare in quantum mechanics. Meanwhile, the condition of exact solvability can be weakened: one may demand that only for a finite part of the spectrum eigenstates and eigenvalues be found explicitly or from a finite algebraic equation. This opens two different possibilities. First, there exist so-called quasi-exactly solvable (QES) systems, whose Hamiltonian can be expressed in terms of the generators of the algebra having a finite-dimensional representation (for one-dimensional potentials the relevant algebra is sl_2 , the corresponding generators representing the effective spin operators) [1–4]. In so doing, the dimension of the finite subspace of the whole Hilbert space is determined by the value of the effective spin that usually enters the QES potential as a parameter. Second, instead of relating the dimension of the finite subspace to an underlying structure of a Lie algebra representation, one may fix the number of known levels ‘by hand’. In the simplest case this number is equal to two, so we deal with two-dimensional subspace. Although such a procedure makes the underlying algebraic structure poorer, it considerably extends the set of potentials with the known part of the spectrum.

Our physical motivation for interest in potentials with two known energy levels stems from the fact that a two-level system represents a very wide class of models often used in solid state and nuclear physics and quantum optics. Let us mention here only a few examples: the Dicke model of interaction between atoms and radiation [5], the Lipkin–Meshkov–Glick model of

interacting nucleons [8] and the phenomenon of macroscopic quantum tunnelling [7]. We would like to stress that it is just the potential description of systems with a finite number of energy levels that enabled one to give a clear and simple explanation of the phenomenon of spin tunnelling [6]. Therefore, finding potentials that correspond to a fixed numbers of eigenstates, was an important step in the calculation of the tunnelling rates in ferro- and super-paramagnets.

Meanwhile, there is also the inner motivation that stems from quantum mechanics as such. From a general viewpoint, recovering potentials from a known set of eigenvalues is nothing other than the reduced variant of the inverse scattering problem. As is well known, using the Darboux transformation, one can get many-soliton solutions of the Schrödinger equation with N energy levels fixed in advance. Understanding how the truncation of the scattering data modifies the structure of the theory, could gain further insight into the inverse scattering approach. The first necessary step here is to find the full solution to the problem for $N = 2$.

If $N = 1$ (only one level is fixed), it follows from the Schrödinger equation that the potential is $U = E + \psi''/\psi$, where E is the value of energy, and ψ is a wavefunction. Choosing any $\psi(x)$ having no zeros at the real axis, we immediately obtain the corresponding potential $U(x)$, regular on the real axis. We would like to stress, however, that, in contrast to the $N = 1$ case, when the solution of the problem is straightforward, even for $N = 2$ resolving this problem required the elaboration of different approaches discussed in the literature. The existence of exact solutions with two levels for power-like potentials was indicated in [9, 10]. The rather powerful technique based on supersymmetric (SUSY) quantum mechanics (see the review [11]) was suggested in [12, 13]. It enables one to generate the potentials with known ground and first excited states. The aim of this paper is to suggest a general approach to the potentials with two known levels valid for any n th excited states. The corresponding method and results turn out to be surprisingly simple and do not require sophisticated techniques (such as, for instance, SUSY quantum mechanics).

2. Basic equations

Consider the Schrödinger equation with the Hamiltonian $H = -\frac{d^2}{dx^2} + U(x)$. Let ψ_1 and ψ_2 be wavefunctions obeying the Schrödinger equations

$$H\psi_1 = E_1\psi_1 \quad (1)$$

$$H\psi_2 = E_2\psi_2. \quad (2)$$

Then it follows from (1), (2) that

$$U = E_1 + \frac{\psi_1''}{\psi_1} \quad (3)$$

$$\frac{\psi_2''}{\psi_2} = E_1 - E_2 + \frac{\psi_1''}{\psi_1}. \quad (4)$$

Let, by definition,

$$\psi_2 = \xi\psi_1. \quad (5)$$

Then we have for ψ_2 from (4):

$$\frac{\psi_1'}{\psi_1} \equiv -\chi' = -\frac{(\xi'' + \Delta E\xi)}{2\xi'} \quad (6)$$

where $\Delta E = E_2 - E_1$. By substitution of (5) and (6) into (4), we obtain three equivalent forms for the potential:

$$U = E_1 - \frac{\Delta E}{2} + \frac{3}{4} \left(\frac{\xi''}{\xi'} \right)^2 - \frac{1}{2} \frac{\xi'''}{\xi'} + \Delta E \frac{\xi\xi''}{\xi'^2} + \frac{1}{4} (\Delta E)^2 \left(\frac{\xi}{\xi'} \right)^2 \quad (7)$$

$$U = E_1 - \frac{\Delta E}{2} \left(1 - 2 \frac{\xi \xi''}{\xi'^2} \right) + \frac{1}{4} (\Delta E)^2 \left(\frac{\xi}{\xi'} \right)^2 - \frac{1}{2} [\xi]_x \tag{8}$$

$$U = E_1 + \chi'^2 - \chi'' \tag{9}$$

where $[\xi]_x \equiv \frac{\xi'''}{\xi'} - \frac{3}{2} \left(\frac{\xi''}{\xi'} \right)^2$ is a Schwarzian derivative (see, e.g., chapter 2.7 of [14]). The wavefunctions of the states under discussion are

$$\psi_1 = e^{-\chi} \quad \psi_2 = e^{-\chi} \xi. \tag{10}$$

Equation (7) gives us the general formula for the potential with two given energy levels. It is expressed directly in terms of their values E_1, E_2 as parameters and one function $\xi(x)$, corresponding wavefunctions are given by (6), (10) and expressed in terms of the same quantities. It is worth noting that the function $\xi(x)$ does not enter the set of known data—rather, the freedom in its choice reflects the fact that for two given eigenvalues there exists an infinite number of potentials having two fixed eigenvalues. The structure of these potentials is not arbitrary but is governed by the form of $\xi(x)$ according to (7).

Equations (6),(7) and(10) constitute the main result of this paper. It is worth stressing that the derivation of equation (7) is very simple, direct and does not need sophisticated technique, such as SUSY machinery. On the other hand, the potential in terms of the function χ' has the form (9), typical for SUSY quantum mechanics, automatically. In so doing, χ' plays the role of a superpotential. As is well known (see, e.g., [11]), one-dimensional quantum mechanics can always be formulated in a SUSY way. However, given a potential, the superpotential cannot be found explicitly for a generic model. Meanwhile, in our case we found not only the potential but also the explicit expression for the superpotential (6).

It is worth stressing that the derivation of (7)–(9) relies strongly on the successful choice of the function $\xi(x)$ that parametrizes the family of solutions. The fact that, for given E_1, E_2 the ratio of two eigenfunctions determines the potential completely generalizes the observation made in [12, 13] for the particular case when the eigenfunctions under consideration refer to the ground and first excited states.

The formalism elaborated above for the one-dimensional Schrödinger equation can be also applied to the three-dimensional one for a particle moving in a spherically symmetrical potential $U(r)$. After the separation of variables, the effective potential entering the radial part of the Schrödinger equation is equal to $U_{ef} = U + \frac{l(l+1)}{r^2}$. Then, repeating calculations step by step, we obtain

$$U = U^{(0)} + \lambda^2 \frac{\xi^2}{4r^4 \xi'} - \frac{\lambda \xi}{r^2 \xi'} \left(\frac{1}{r} + \frac{\xi''}{\xi} \right) - \lambda \frac{\Delta E \xi^2}{2r^2 \xi'^2} + \frac{\lambda - 2l_1(l_1 + 1)}{2r^2} \tag{11}$$

$$\lambda = (l_2 - l_1)(1 + l_1 + l_2) \tag{12}$$

and $U^{(0)}$ is expressed in terms of E_1, E_2 and ξ using the same formulae (7)–(9) as in the one-dimensional case.

It is worth noting that now a new interesting possibility can arise that is absent in the one-dimensional case: $\Delta E = 0$. This becomes possible due to the fact that two quantum states can refer to different effective potentials ($l_1 \neq l_2$): we are faced with degeneracy with respect to the angular momentum. Then the potential acquires the form

$$U = E_1 - \frac{1}{2} [\xi]_r + \lambda^2 \frac{\xi^2}{4r^4 \xi'} - \frac{\lambda \xi}{r^2 \xi'} \left(\frac{1}{r} + \frac{\xi''}{\xi} \right) + \frac{\lambda - 2l_1(l_1 + 1)}{2r^2}. \tag{13}$$

We will not discuss the three-dimensional case further and will concentrate on the one-dimensional one.

3. General properties and classification of states

The potential $U \equiv U(E_1, E_2, \xi)$ possesses the symmetries that follow directly from (7):

$$\begin{aligned} U(E_1, E_2, a\xi) &= U(E_1, E_2, \xi) \\ U(E_1, E_2, \xi) &= U(E_2, E_1, \xi^{-1}). \end{aligned} \quad (14)$$

Throughout the paper we assume that the potential $U(x)$ is regular everywhere, except, perhaps, at infinity. Then all zeros and poles of the function $\xi(x)$ are simple—otherwise the potential U would become singular and the wavefunction ψ_2 would cease to be normalizable. If the function ξ has a pole at $x = x_1$, $\xi \approx A(x - x_1)^{-1}$, one gets from (6) that $\chi' \approx -(x - x_1)^{-1}$, so $\psi_1(x_1) = 0$, $\psi_2(x_1) = \text{const} \neq 0$. Therefore, every zero of ξ generates a node of the wavefunction ψ_2 and every pole of ξ generates a node of ψ_1 .

The set of possible nodes of wavefunctions also depends on the behaviour of the function $\chi(x)$ in the vicinity of zeros of the function $\xi'(x)$ due to possible zeros of the factor $\exp(-\chi)$ in (10). Let $\xi'(x_0) = 0$. Then, according to (6), if $x \rightarrow x_0$, $\chi' \approx B/(x - x_0)$, where $B = (\xi'' + \Delta E \xi)|_{x=x_0}/2\xi''(x_0)$, and the potential contains the term that behaves like $B(B + 1)(x - x_0)^{-2}$. The regularity of the potential entails $B = 0$ or -1 . Consider these two cases separately.

First, let $B = 0$. Now the condition

$$(\xi'' + \Delta E \xi)|_{x=x_0} = 0 \quad (15)$$

must hold. In so doing, the function $\chi(x)$ is regular in the vicinity of x_0 due to the condition (15) and the factor $\exp(-\chi)$ cannot vanish.

Consider the case $B = -1$. Now we have

$$(3\xi'' + \Delta E \xi)|_{x=x_0} = 0. \quad (16)$$

Then $\chi' \approx -(x - x_0)^{-1}$ and the functions ψ_1 and ψ_2 share the common node at $x = x_0$ due to the factor $\exp(-\chi)$, as it follows from (10). (For example, if the potential is even, $U(-x) = U(x)$, all wavefunctions of odd states vanish at $x = 0$.)

As a result, we arrive at the conclusion that, if (i) $\xi(x)$ has n_1 poles and n_2 zeros, (ii) $\xi'(x)$ has $m^{(0)}$ zeros such that (15) is satisfied ($B = 0$) and $m^{(-)}$ zeros such that (16) is satisfied ($B = -1$), the function $\psi_1(x)$ describes the state with the number of nodes $N_1 = n_1 + m^{(-)}$, while $\psi_2(x)$ corresponds to the state with the number of nodes $N_2 = n_2 + m^{(-)}$. Therefore, the quantum number that labels states is equal to N_1 for ψ_1 and N_2 for ψ_2 ($N_{1,2} = 0, 1, 2, \dots$).

It follows directly from definition (5): if ψ_1 has simple zeros at x_i and ψ_2 has simple zeros at x_k with $x_i \neq x_k$, the function ξ has poles at $x = x_i$ and zeros at $x = x_k$. However, if some $x_i = x_k$, corresponding zeros of both functions compensate each other and this results in the fact that, if some coefficients $B = -1$, the state labels are not determined completely by the numbers n_1, n_2 .

Let us have two fixed energy levels E_1, E_2 ($E_2 > E_1$) and the function $\xi(x)$ such that $N_2 > N_1$. Then, it is the potential $U(E_1, E_2, \xi)$ for which the level E_1 belongs to the N_1 th state and E_2 corresponds to the N_2 th state. If $N_1 > N_2$, the relevant potential is $U(E_1, E_2, \xi^{-1}) = U(E_2, E_1, \xi)$, the level E_1 belongs to the N_2 th state, while E_2 corresponds to the N_1 th state. If $N_1 = N_2$, the function $\xi(x)$ is not suitable for constructing $U(x)$ with two different fixed levels. Indeed, in this case we would have had two different wavefunctions with the same number of zeros corresponding to two different levels, in disagreement with the oscillation theorem.

4. Comparison with the SUSY approach and Tkachuk's results

Let us introduce the function W_+ according to

$$W_+ = \frac{\delta \xi}{\xi'} \tag{17}$$

where $\delta = \Delta E > 0$.

Then, with (6) taken into account, we obtain

$$\begin{aligned} \chi' &= \frac{1}{2} \left(W_+ - \frac{W'_+ - \delta}{W_+} \right) \equiv W \equiv \frac{W_+ - W_-}{2} & U &= E_1 + W^2 - W' \\ \psi_1 &= e^{-\int dx W} & \psi_2 &= \xi e^{-\int dx W} = W_+ \exp \left[-\frac{1}{2} \int dx (W_+ + W_-) \right] \end{aligned} \tag{18}$$

where, by definition,

$$W_- = \frac{W'_+ - \delta}{W_+}. \tag{19}$$

Since ψ_1 must be normalizable, $\text{sign}(W_+(\pm\infty)) = \pm 1$. Let W_+ have only one zero at $x = x_0$. If we want W_- to be regular at $x = x_0$, $W'_+(x_0) = \delta$.

The formulae (18) (with $E_1 = 0$ and $\delta = 2\varepsilon$) were derived in [12] by solving equations for the superpotential which appear in the SUSY approach. In our terms, this approach deals with the function $\xi(x)$, such that ξ has only one zero (just in the point x_0), ξ' changes sign nowhere and ξ does not have poles on a real axis (otherwise they would give rise to additional zeros of W_+). Therefore, in the situation considered in [12, 13] ψ_1 corresponds to the ground state and ψ_2 describes the first excited state—in agreement with the conclusion of the previous section of this paper. Thus, in this particular case our approach reproduces the results of [12, 13].

5. Illustrations. Deformations of potential leaving two levels fixed

To illustrate the general results (7)–(9), let us consider the following example: $\xi = x^4 + 2x^2x_0^2 - x_1^4$. The derivative $\xi' = 0$ at $x = 0$; therefore, as is explained in the preceding section, the corresponding example cannot belong to the set considered in [12]. After straightforward calculations, one obtains

$$U = E_1 + \frac{x^2x_0^4}{4x_1^8} + \frac{1}{4x_1^4} \left[A_0 + \frac{A_1}{x^2 + x_0^2} + \frac{A_2}{(x^2 + x_0^2)^2} \right] \tag{20}$$

where

$$\begin{aligned} A_0 &= 2x_0^2 \left(2 + \frac{x_0^4}{x_1^4} \right) & A_1 &= (3x_1^4 + x_0^4) \left(5 - \frac{x_0^4}{x_1^4} \right) \\ A_2 &= -(3x_1^4 + x_0^4)x_0^2 \left(7 + \frac{x_0^2}{x_1^4} \right). \end{aligned} \tag{21}$$

The functions are equal to

$$\psi_1 = (x^2 + x_0^2)^{-\alpha} \exp \left(-\frac{x^2x_0^2}{2x_1^4} \right) \quad \alpha = \frac{3x_1^4 + x_0^4}{4x_1^4} \tag{22}$$

$$\psi_2 = \psi_1(x^2 - x_-^2)(x^2 + x_+^2) \quad x_{\pm} = \sqrt{x_0^4 + x_1^4 \pm x_0^2}. \tag{23}$$

It is seen from (22), (23) that ψ_1 has no nodes at the real axis, while ψ_2 vanish at $x = \pm x_-$. Therefore, ψ_1 corresponds to the ground state, while ψ_2 describes the second excited state.

As we see from (8), the Schwarzian derivative is an essential ingredient of the expression for the potential under discussion. It is known that the Schwarzian derivative is invariant with respect to the linear-fractional transformations. Therefore, it is instructive to apply such a transformation to the potential as a whole and look at the resulting expression. Let us make the substitution

$$\xi = \frac{c_2\eta + d_2}{c_1\eta + d_1}. \quad (24)$$

We will use it below for generating in an explicit form rather rich families of the potentials, corresponding to two known levels. As $[\eta]_x$ remains invariant, only the part of (8) which contains the terms proportional to ΔE and $(\Delta E)^2$ changes under this transformation. Then the potential and wavefunctions of the states under discussion take the form

$$U = E_1 - \frac{\Delta E}{2} + \frac{2\Delta E c_1(d_2 + c_2\eta)}{c_1d_2 - c_2d_1} + \frac{1}{4} \frac{Y^2}{\eta^2} - \frac{\eta''}{\eta^2} Y - \frac{1}{2} [\eta]_x \quad (25)$$

$$Y = \Delta E \frac{(c_1\eta + d_1)(c_2\eta + d_2)}{c_1d_2 - c_2d_1}.$$

$$\psi_{1,2} = e^{-\rho} \Phi_{1,2} \quad \Phi_{1,2} = c_{1,2}\eta + d_{1,2} \quad (26)$$

$$\rho' = \frac{\eta'' - Y}{2\eta'} \quad \chi = \rho - \ln(c_1\eta + d_1).$$

If $c_2 = 0 = d_1$, one can see that $Y = \Delta E\eta$ and $U(E_1, E_2, \xi) = U(E_1, E_2, \eta^{-1}) = U(E_2, E_1, \eta)$ in accordance with (14).

In the limit

$$c_1 = 0 = d_2 \quad (27)$$

we obtain the original potential $U(E_1, E_2, \xi) = U(E_1, E_2, \eta)$.

Let us assume first we have some function $\eta(x)$ characterized by the set of numbers $(n_1, n_2, m^{(-)})$ introduced in section 3. The original potential has the form (7) with $\xi = \eta$. Then, let us take $\xi(x)$ according to (24) with nonzero arbitrary coefficients c_i and d_i . As the result of the transformation of (24), each of the aforementioned numbers can change (for example, zeros $x_k^{(1)}$ of the combination $c_1\eta + d_1$ generate poles of ξ , zeros $x_i^{(2)}$ of $c_2\eta + d_2$ correspond to zeros of ξ , each zero $x_j^{(0)} \neq x_k^{(1)}$ of η' generates a zero of ξ'). Therefore, the levels E_1, E_2 which corresponded to the N_1 th and N_2 th levels now can, in principle, correspond to other quantum numbers (M_1, M_2) . Making the transformation, inverse to (24), one may restore the form of the potential (7), but now $\xi \neq \eta$, with ξ having the form (24), in which coefficients under discussion play the role of parameters. Thus, we obtain a family of deformations leaving two energy levels E_1, E_2 fixed. These deformations can be described, on an equal footing, by the deformation of the form of the function $\xi(x)$ or of that of the potential.

For definiteness, we will choose the second possibility. If $c_1, c_2 \neq 0$, one can always achieve $c_1 = c_2 \equiv c$ by proper rescaling of the function $\xi(x)$ that does not affect, according to (14), the function $U(x)$. Then, defining

$$c = 2\beta \quad d_1 = -\bar{\delta} - \Delta E \quad d_2 = -\bar{\delta} + \Delta E \quad \gamma = \frac{(\Delta E)^2 - \bar{\delta}^2}{4\beta} \quad (28)$$

we obtain

$$Y = \beta\eta^2 - \bar{\delta}\eta - \gamma \quad (29)$$

$$U = E_1 + \frac{\Delta E}{2} - \bar{\delta} + 2\beta\eta - \frac{1}{2} [\eta]_x + \frac{1}{4} \frac{(\beta\eta^2 - \bar{\delta}\eta - \gamma)^2}{\eta^2} - \frac{\eta''}{\eta^2} (\beta\eta^2 - \bar{\delta}\eta - \gamma). \quad (30)$$

Below we will see how introducing nonzero parameters β, γ affects the potential and wavefunctions (26).

5.1. Example 1

Let us choose η as a polynomial:

$$\begin{aligned} \eta' &= 4ax(x_0^2 - x^2) & a > 0 \\ \eta &= a(V_0 + x_0^4 - z^2) & z = x^2 - x_0^2 & V_0 = \text{const.} \end{aligned} \tag{31}$$

Demanding that ρ' be regular at $x = 0$ and at $x = \pm x_0$, we obtain from (26) the constraints

$$\gamma = \beta a^2 (V_0 + x_0^4)^2 + 8ax_0^2 - \delta a (V_0 + x_0^4) = \beta a^2 V_0^2 - 4ax_0^2 - \bar{\delta} a V_0 \tag{32}$$

whence

$$\begin{aligned} V_0 &= -\frac{x_0^4}{2} - \frac{6}{a\beta x_0^2} + \frac{\bar{\delta}}{2\beta a} \\ \gamma &= \beta^{-1} \left(R - \frac{\bar{\delta}^2}{4} \right) & R &= \frac{(\Delta E)^2}{4} = \frac{\beta^2 a^2 x_0^8}{4} + 2\beta a x_0^2 + 36x_0^{-4}. \end{aligned} \tag{33}$$

The expression for the function ξ reads

$$\xi = 1 + \frac{2\Delta E}{A_2 x^4 + A_1 x^2 + A_0}. \tag{34}$$

The potential can be obtained from (30) or directly from (7). It has the form $U = \sum_{n=0}^5 c_{2n} x^{2n}$, $c_{10} = (\beta a)^2/64$. It is convenient to rescale the variable in such a way that the coefficient in the potential U at the largest power be equal to 1. This can be achieved by $x = \lambda y$, $\beta a \lambda^6 = 8\omega$, where $\omega = 1$ or -1 . After some manipulations we get the new potential $\bar{U} = \lambda^2 U$, corresponding to levels $\bar{E}_{1,2} = \lambda^2 E_{1,2}$:

$$\begin{aligned} \bar{U} &= y^{10} - 6\mu y^8 + \left(13\mu^2 + \frac{3\omega}{\mu} \right) y^6 - (12\mu^3 + 22\omega) y^4 + \left(4\mu^4 + 31\mu\omega + \frac{9}{4\mu^2} \right) y^2 \\ &\quad + \frac{\bar{E}_1 + \bar{E}_2}{2} - \frac{15}{2\mu} - 6\omega\mu^2 \end{aligned} \tag{35}$$

$$\rho = \frac{y^6}{6} - \frac{3\mu\omega y^4}{4} + y^2 \left(\mu^2\omega + \frac{3}{4\mu} \right) \quad (\Delta \bar{E})^2 = 16\mu^{-2}(4\mu^6 + 4\omega\mu^3 + 9) \tag{36}$$

where $\mu = x_0^2/\lambda^2$. (In fact, the formula (35) can also be extended to negative μ .) The quantities μ and $\bar{E}_{1,2}$ are not independent but connected by equation (36) that appeared due to the condition of the regularity of the potential (15). It is worth noting that the parameter $\bar{\delta}$ cancels and does not enter the expression (35) due to the conditions (32), (33).

It follows from (26), (28) and (33) that, up to the constant factor, the function $\Phi_{1,2} = z^2 - 2\mu z + q_{1,2} = (z - z_1)(z - z_2)$, $z \equiv y^2$, $z_{1,2} = \mu \pm \sqrt{\mu^2 - q_{1,2}}$, $q_{1,2} = \frac{\mu^2}{2} + \frac{3}{4\mu}\omega \pm \frac{\Delta \bar{E}}{16}\omega$. Let, for definiteness, $E_2 > E_1$. Consider first the case $\omega = 1$. Then after some algebra one easily finds that $0 < q_2 < \mu^2$ and $q_1 > \mu^2$. Therefore, the function Φ_1 does not have the nodes at the real axis and corresponds to the ground state. The function Φ_2 has four zeros and corresponds to the fourth state. In a similar way, we obtain that for $\omega = -1$ the quantities $q_1 < 0$, $0 < q_2 < \mu^2$, so the wavefunctions under discussion describe the second and fourth excited states.

5.2. Example 2

One may exploit the ansatz (17) with $\xi = \eta$ for the potential (30) with $\beta, \gamma \neq 0$. Substituting it into (26), one obtains

$$\rho' = -\frac{1}{2} \left[\frac{(W'_+ - \delta)}{W_+} - \frac{\bar{\delta} W_+}{\delta} + \frac{\beta W_+}{\delta} \exp\left(\delta \int \frac{dx}{W_+}\right) - \frac{\gamma W_+}{\delta} \exp\left(-\delta \int \frac{dx}{W_+}\right) \right]. \tag{37}$$

Let us consider the example

$$W_+ = ax + bx^3 \quad \gamma = 0 \quad \beta \neq 0 \quad a = \delta = \bar{\delta} > 0 \quad b > 0. \tag{38}$$

After simple but cumbersome manipulations we get the potential

$$U = \frac{\beta x}{x_0} \left(-\frac{x_0^2}{r} + 3r + \frac{ar^3}{2} - \frac{ar^5}{2x_0^2} \right) + u \quad r \equiv \sqrt{x_0^2 + x^2} \quad x_0^2 \equiv \frac{a}{b} \tag{39}$$

$$u = \frac{b^2}{4}x^6 + \left(\frac{ab}{2} + \frac{\beta^2}{4} \right)x^4 + \frac{(a^2 - 12b)}{4}x^2 - \frac{a}{2} + \frac{3}{4(x_0^2 + x^2)} + \frac{3}{4} \frac{x_0^2}{(x^2 + x_0^2)^2}$$

and wavefunctions

$$\psi_1 = \frac{(x^2 + x_0^2)^{3/4}}{\left(x + \sqrt{x^2 + x_0^2} \right)^{\beta x_0^3/16}} e^{-\alpha} \left(1 - \frac{\beta}{a} \eta \right) \quad \psi_2 = \frac{\eta(x^2 + x_0^2)^{3/4}}{\left(x + \sqrt{x^2 + x_0^2} \right)^{\beta x_0^3/16}} e^{-\alpha} \tag{40}$$

$$\eta = \frac{xx_0}{\sqrt{x^2 + x_0^2}} \quad \alpha = \frac{ax^2}{4} + \frac{bx^4}{8} - \frac{\beta x}{16x_0} (2x^2 + x_0^2) \sqrt{x^2 + x_0^2}.$$

It is seen from (40) that $\eta' > 0$ and the potential (39) is regular for any choice of parameters, so the conditions (15), (16) are irrelevant for this case. The functions ψ_1 and ψ_2 are normalizable, provided $\beta^2 < ab$. It can be readily seen from (40) that the function ψ_2 has one node at $x = 0$, whereas ψ_1 vanish nowhere. Thus, ψ_1 and ψ_2 correspond to the ground and the first excited states, respectively. In the limit $\beta = 0$ we reproduce the result for example 3 of [12].

5.3. Example 3

Let now $W_+ = A(\text{sh}x - \text{sh}x_0)$, $\gamma = 0$, $\beta \neq 0$, $\delta = \bar{\delta}$. Then, repeating calculations for this case, we get

$$U = \frac{E_1 + E_2}{2} - \frac{\delta}{2} + U_0(x) + \frac{\beta \text{sh} \frac{(x-x_0)}{2}}{\text{ch}x_0 \text{ch} \frac{(x+x_0)}{2}} \left[\text{ch}x + \text{ch}x_0 - \frac{\delta(\text{sh}x - \text{sh}x_0)^2}{2\text{ch}x_0} \right] + \frac{\beta^2 \text{sh}^4 \frac{(x-x_0)}{2}}{\text{ch}^2 x_0}$$

$$U_0(x) = \frac{\delta^2}{4\text{ch}^2 x_0} (\text{sh}x - \text{sh}x_0)^2 - \frac{\delta}{2\text{ch}x_0} (2\text{ch}x - \text{ch}x_0) + \frac{1}{4} \tag{41}$$

$$\psi_1 = \text{ch} \frac{(x+x_0)}{2} e^{-\alpha} \left(1 - \frac{\beta}{\delta} \eta \right) \quad \psi_2 = \text{ch} \frac{(x+x_0)}{2} e^{-\alpha} \eta \quad \eta = \frac{\text{sh} \frac{(x-x_0)}{2}}{\text{ch} \frac{(x+x_0)}{2}}$$

$$\alpha = (2\text{ch}x_0)^{-1} [\delta \text{ch}x - \beta \text{sh}(x - x_0) + x(\beta - \delta \text{sh}x_0)].$$

Here U_0 is the potential corresponding to the spin- $\frac{1}{2}$ anisotropic paramagnet in an oblique magnetic field [1]. The function $\eta' > 0$, so $\rho(x)$ is regular in any point for any choice of parameters. The wavefunction is normalizable provided $-\delta e^{-x_0} < \beta < \delta e^{x_0}$. One can easily show that it follows from this condition that ψ_1 does not have nodes and corresponds to the ground state, while ψ_2 has one node at $x = x_0$ and corresponds to the first excited state. In the limit $\beta = 0$ the example 1 of [12] is reproduced.

6. Concluding remarks

Thus, in a very simple and direct approach we found a rather general solution that gives us the structure of potentials with two known eigenstates E_1, E_2 in terms of one function $\xi(x)$ and one parameter coinciding with the energy difference $\Delta E = E_2 - E_1$. Moreover, we get not only the potential itself, but also (in terms of the SUSY language) the superpotential. Depending

on properties of the function $\xi(x)$ and the type of the regularity condition of the potential in the vicinity of zeros of $\xi'(x)$ ((15) or (16)), one can obtain not only the ground or first excited state but, in principle, any pair of levels. The natural question arises whether the approach of this paper is extendable to the case of three (or more) levels. This problem needs separate treatment. We hope that movement in this direction will promote further understanding links between QES-type systems, SUSY and the inverse scattering approaches.

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